# MORE EXAMPLES

BRIAN CONRAD

Some important examples and explications of Grothendieck duality which should have been included in Chapter 5 of [C1] were overlooked. Several such topics are treated here. One question which can be asked about the trace map $\mathrm{H}^n(X, \Omega^n_{X/k}) \to k$ for a proper smooth scheme $X$ of pure dimension $n$ over a field $k$ is how it relates to the trace map for a smooth divisor. Generalizing a little, one can ask whether there is an analogous statement if one replace "smooth" with "CM". This matter is addressed in the relative case in §1, and the answer winds up involving a sign of $(-1)^{n-1}$. In a different direction, one can ask how the Grothendieck trace map for a proper smooth variety over $\mathbf{C}$ is related to a "topological" trace map defined in terms of topological Poincaré duality, as in [C1, (2.3.4)]. They do not agree, but instead are off by a sign of $(-1)^n$. In the projective case, one can analyze this problem quite pleasantly by using Bertini's theorem, induction, and the divisor compatibility treated in §1, once one has also worked out the analytic analogue of §1. We work out such an analogue in §2, and the answer winds up involving a sign of $(-1)^n$, a contrast with the sign of $(-1)^{n-1}$ in the algebraic case, and it is this discrepancy of $-1$ that ultimately leads to the sign of $(-1)^n$ when comparing the Grothendieck and topological traces in the projective algebraic case (since each of the $n$ hyperplane slices contributes a sign of $-1$ in the comparison, but everything agrees in dimension 0).

Despite naive impressions, our sign of $(-1)^n$ in the analytic/algebraic trace comparison *is* consistent with the lack of such a sign in the analytic/algebraic comparison given in the appendix to [D]. The explanation for the consistency is given in the pp. 33, 36 remarks in [C2]. The analytic/algebraic comparison in the general proper case can be reduced to the projective case (and even just the case of projective spaces!) by using local cohomology, as we explain in §3. Finally, in §4 we work out the comparison between the residue map and the Chern class map for line bundles on curves (where the Chern class is viewed with values in the ground field rather than in $\mathbf{Z}$).

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## 1. Trace maps and coboundaries: scheme case

Consider a commutative diagram of schemes

$$\begin{array}{ccc} D & \xrightarrow{j} & X \\ & \searrow^{g} & \downarrow{\scriptstyle f} \\ & & S \end{array}$$

in which $f$ is proper CM of pure relative dimension $n > 0$ and $j$ is a closed immersion which is transversally regular relative to $S$ of pure relative codimension 1, so $g$ is proper CM of pure relative dimension $n - 1 \geq 0$. Let $\omega_f$ and $\omega_g$ denote the respective relative dualizing sheaves for $f$ and $g$. There are canonical trace morphisms

$$\gamma_g : \mathrm{R}^{n-1} g_*(\omega_g) \to \mathscr{O}_S, \quad \gamma_f : \mathrm{R}^n f_*(\omega_f) \to \mathscr{O}_S$$

which are of formation compatible with arbitrary base change, and likewise there are canonical base change compatible isomorphisms $\omega_g \simeq \Omega^{n-1}_{D/S}$, $\omega_f \simeq \Omega^n_{X/S}$ when $g$ and $f$ are smooth. Using [C1, (3.6.11)], there is a

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canonical isomorphism of $\mathscr{O}_D$-modules

$$\omega_g \simeq \mathscr{E}xt^1_{\mathscr{O}_X}(j_*\mathscr{O}_D, \omega_f)$$

which is of formation compatible with base change on $S$. Moreover, using the fundamental local isomorphism from [C1, (2.5.1)], there is a canonical isomorphism of $\mathscr{O}_D$-modules

$$\mathscr{E}xt^1_{\mathscr{O}_X}(j_*\mathscr{O}_D, \omega_f) \simeq \omega_f(D)|_D$$

which is compatible with arbitrary base change on $S$. Putting these together, we get a canonical isomorphism of $\mathscr{O}_D$-modules

$$\omega_f(D)|_D \simeq \omega_g \tag{1.1}$$

which is compatible with arbitrary base change on $S$ and for smooth $f$ and $g$ recovers exactly the classical Koszul isomorphism $\zeta'_{j,f}$ from [C1, case ($c$), pp.29–30]. This explicit description of (1.1) in the smooth case rests on the fact that [C1, (3.6.11)] recovers [C1, (3.5.7)] in the smooth case and that [C1, Lemma 3.5.3] should have *no* sign (this lack of sign was observed by Gabber: the mistake in the proof is that $\mathrm{d}t \wedge \mathrm{d}x$ in [C1, line 12, p. 164] should be $\mathrm{d}x \wedge \mathrm{d}t$, thereby contributing another sign of $(-1)^{n(N-n)}$ to leave no sign in the final result).

Using (1.1), we arrive at a canonical short exact sequence

$$0 \to \omega_f \to \omega_f(D) \to j_*\omega_g \to 0 \tag{1.2}$$

which is compatible with base change on $S$ and is exactly the classical such sequence in terms of sheaves of top degree relative differential forms in the smooth case, with the right map locally described by

$$(\mathrm{d}t/t) \wedge \eta \mapsto j_*(\eta|_D) \tag{1.3}$$

It now makes sense to consider the diagram

$$\begin{array}{ccc} \mathrm{R}^{n-1}g_*(\omega_g) & \xrightarrow{\simeq} & \mathrm{R}^{n-1}f_*(j_*\omega_g) \\ {\scriptstyle\gamma_g}\downarrow & & \downarrow{\scriptstyle\delta} \\ \mathscr{O}_S & \xleftarrow[\gamma_f]{} & \mathrm{R}^n f_*(\omega_f) \end{array} \tag{1.4}$$

in which $\delta$ is the connecting homomorphism in the long exact cohomology sequence attached to (1.2). The theorem we wish to prove is:

**Theorem 1.1.** *The diagram* (1.4) *is* $(-1)^{n-1}$*-commutative.*

Before giving the general proof, we consider a concrete example with $n = 1$. If $X$ is a proper smooth connected curve over an algebraically closed field $k$ and if $D = \{x\}$ is a closed point of $X$, then there is a canonical short exact sequence

$$0 \to \Omega^1_{X/k} \to \Omega^1_{X/k}(x) \to k(x) \to 0$$

in which $k(x)$ denotes the structure sheaf of the reduced closed subscheme $\{x\}$ and the map to $k(x)$ is the residue map at $x$. By [C1, Thm B.2.2], the Grothendieck trace map $\mathrm{H}^1(X, \Omega^1_{X/k}) \to k$ is the *negative* of the classical residue map (defined by computing the sheaf cohomology of $\Omega^1_{X/k}$ in terms of the classical two-term flasque resolution). Thus, in this case the theorem asserts that the composite map

$$k = \mathrm{H}^0(X, k(x)) \to \mathrm{H}^1(X, \Omega^1_{X/k}) \to k,$$

with the second map the residue map, is multiplication by $-1$. This can be checked by hand, using a Čech cohomology calculation as in [C1, p. 289]. Now we prove the general case.

*Proof.* Before we begin the proof, we discuss a special case. Suppose $S = \operatorname{Spec} \mathbf{Z}$, $D = \mathbf{P}^{n-1}_{\mathbf{Z}}$, $X = \mathbf{P}^n_{\mathbf{Z}}$, and $j : D \hookrightarrow X$ is the map $[t_0, \dots, t_{n-1}] \mapsto [t_0, \dots, t_{n-1}, 0]$. In this case, one can use the explicit Čech cohomology description of the Grothendieck trace on projective space [C1, (2.3.3), Lemma 3.4.3(TRA3), (3.4.13)] and an explicit coboundary computation in Čech cohomology to easily compute directly that (1.4) commutes up to $(-1)^{n-1}$ in this special case. This calculation of course rests on the fact that in the smooth case the sequence (1.2) is exactly the classical sequence one expects in terms of differential forms, resting on (1.3). With the sign determined in this special case, it now suffices to prove the general case up to an abstract undetermined universal sign depending *only* on $n$ (and not on $S$, $f$, $g$, or $j$). The advantage of this is that when analyzing various diagrams in derived categories, we can freely move translation functors through total derived functors without too much worry and in general can ignore most formal combinatorics of signs in homological algebra. Such "sloppiness" in what follows will at worst introduce an ambiguity governed by a universal sign depending only on $n$, and hence is harmless for what we need to do. It should be remarked that although we will give a rather abstract-looking proof with derived categories, if we were only interested in the smooth (rather than CM) case then we could give a more explicit and down-to-earth proof by using Lemma 3.5.3 and the concrete interpretation of the connecting homomorphism from Hom to $\mathrm{Ext}^1$ in terms of short exact sequences representing elements of $\mathrm{Ext}^1$.

Note also that everything under consideration, including (1.1) and (1.2), is of formation compatible with base change on $S$, so by standard arguments from EGA $\mathrm{IV}_3$, §8ff, we may assume $S$ is an artin local scheme, so $S$ admits a normalized residual complex $\mathscr{I}[0]$, where $\mathscr{I}$ is the *coherent* sheaf associated to an injective hull of $k(s)$ over the local artin ring $\mathscr{O}_{S,s}$, with $s$ the unique point of $S$. We will now base our analysis on the formulation of duality theory in terms of residual complexes [C1, §3.2ff]. This frees us from the specificity of differential forms and permits a more conceptual (but also quite abstract) argument than in the smooth case alone.

Evaluation of the general isomorphism of $\delta$-functors $g^! \simeq j^! f^!$ on $\mathscr{O}_S[0]$ yields a canonical isomorphism

$$\omega_g[n-1] \simeq j^\flat(\omega_f[n])$$

in $\mathbf{D}(D)$ which, essentially by definition, recovers (1.1) up to universal signs depending only on $n$. This isomorphism fits into the top row of the following commutative diagram in $\mathbf{D}(S)$ which encodes the general transitivity of the derived category trace:

$$\begin{array}{ccc} \mathbf{R}g_*(\omega_g[n-1]) & \xrightarrow{\simeq} & \mathbf{R}f_*\mathbf{R}j_*j^\flat(\omega_f[n]) \\ {\scriptstyle \mathrm{Tr}_g}\downarrow & & \downarrow{\scriptstyle \mathrm{Tr}_j} \\ \mathscr{O}_S & \xleftarrow[\mathrm{Tr}_f]{} & \mathbf{R}f_*(\omega_f[n]) \end{array}$$

After passing to $\mathrm{H}^0$'s, the left and bottom maps recover $\gamma_g$ and $\gamma_f$ respectively, up to a universal sign depending on $n$ (the ambiguity arising from the need to specify certain universal conventions for defining isomorphisms such as $\mathrm{H}^0(C^\bullet[j]) \simeq \mathrm{H}^j(C^\bullet)$ when relating ordinary derived functors to total derived functors). Since such universal signs are of no concern to us, we're reduced to proving that, up to a universal sign depending on $n$, the derived category connecting homomorphism

(1.5) $$\delta : j_*\omega_g[n-1] \to \omega_f[n]$$

arising from the mapping cone of (1.2) coincides with the derived category map

(1.6) $$j_*\omega_g[n-1] \simeq \mathbf{R}j_*j^\flat\omega_f[n] \to \omega_f[n],$$

where the second step here is induced by the derived category trace $\mathrm{Tr}_j$ (see [C1, (1.3.3)ff] for the relationship between classical connecting homomorphisms and mapping cones).

We now formulate everything in terms of residual complexes. Recall from [C1, (3.3.17)] that the functor $j^\Delta$ on residual complexes is $\mathscr{H}om_X(j_*\mathscr{O}_D, \cdot)$, viewed with values in complexes of $\mathscr{O}_D$-modules. We will let $j^\Delta$ denote this functor on complexes of arbitrary $\mathscr{O}_X$-modules, so $j^\flat = \mathbf{R}(j^\Delta)$. Using [C1, (3.3.6)], we have canonical isomorphisms

(1.7) $$\omega_f[n] \simeq \mathscr{H}om^\bullet_X(f^*\mathscr{I}, f^\Delta\mathscr{I})$$

in $\mathbf{D}(X)$ and

$$\omega_g[n-1] \simeq \mathscr{H}om^\bullet_D(g^*\mathscr{I}, g^\Delta\mathscr{I}) \simeq \mathscr{H}om^\bullet_D(j^*f^*\mathscr{I}, j^\Delta f^\Delta\mathscr{I})$$

in $\mathbf{D}(D)$, so we get an isomorphism

$$j_*\omega_g[n-1] \simeq \mathscr{H}om^\bullet_X(f^*\mathscr{I}, j_*j^\Delta f^\Delta \mathscr{I}) \tag{1.8}$$

in $\mathbf{D}(X)$. Note that since $\mathscr{I}$ is coherent, all of these $\mathscr{H}om$-sheaves are quasi-coherent (though we will never need this fact).

The crucial point is that the complex on the right side of the isomorphism (1.7) is concentrated in degrees from $-n$ to 0, and not only gives a flasque resolution of $\omega_f[n]$ but even consists of $j^\Delta$-acyclics (so it is suitable for computing $j^\flat(\omega_f[n])$). To check this acyclicity, we observe that $j^\Delta$ has cohomological dimension 1, so we just have to check that if $t$ is a local equation for $D$ in $X$ then multiplication by $t$ on $\mathscr{H}om^\bullet_X(f^*\mathscr{I}, \mathscr{J})$ is a *surjective* map for any injective $\mathscr{O}_X$-module $\mathscr{J}$. For this, we just need $t$ not to be a zero-divisor on $f^*\mathscr{I}$ (over the open where $t$ is defined). But $\mathscr{I}$ lives on $S$ and

$$0 \to \mathscr{O}_X(-D) \to \mathscr{O}_X \to j_*\mathscr{O}_D \to 0$$

is a short exact sequence with all terms $S$-flat, so it is clear that $t$ is not a zero-divisor on $f^*\mathscr{I}$. As a consequence, with the resolution (1.7) we see that applying $j_*$ to $\omega_g[n-1] \simeq j^\flat(\omega_f[n])$ yields an isomorphism which explicates to be exactly (1.8). This is very important,

Using [C1, Thm 3.4.1(3), Lemma 3.4.3(TRA2)] and the canonical inclusion $j_*j^\Delta \to \mathrm{id}$, we get a commutative diagram in $\mathbf{D}(X)$

$$\begin{array}{ccccc}
j_*\omega_g[n-1] & \xrightarrow{\simeq} & \mathbf{R}j_*j^\flat\omega_f[n] & \xrightarrow{\mathrm{Tr}_j} & \omega_f[n] \\
 & \searrow^{\simeq} & & & \downarrow \simeq \\
 & & \mathscr{H}om^\bullet_X(f^*\mathscr{I}, j_*j^\Delta f^\Delta\mathscr{I}) & \longrightarrow & \mathscr{H}om^\bullet_X(f^*\mathscr{I}, f^\Delta\mathscr{I})
\end{array}$$

in which the top row is (1.6) and the left column is (1.8). Since we need to identify (1.6) with (1.5), we are reduced to proving the commutativity in $\mathbf{D}(X)$ of the diagram

$$\begin{array}{ccc}
j_*\omega_g[n-1] & \xrightarrow{\delta} & \omega_f[n] \\
\simeq \downarrow & & \downarrow \simeq \\
\mathscr{H}om^\bullet_X(f^*\mathscr{I}, j_*j^\Delta f^\Delta\mathscr{I}) & \longrightarrow & \mathscr{H}om^\bullet_X(f^*\mathscr{I}, f^\Delta\mathscr{I})
\end{array} \tag{1.9}$$

at least up to a universal sign depending only on $n$, where the bottom row is the canonical map of complexes, the top row arises from the mapping cone construction, and the left column is (1.8), which we have noted essentially amounts to the coupling of the "definition" $\omega_g = \mathscr{E}xt^1_X(\mathscr{O}_D, \omega_f)$ and the $j^\Delta$-acyclic resolution (1.7). Now the trick is to carefully explicate $\delta$ in terms of a mapping cone complex involving terms as in the bottom row of the diagram. Once this is done, everything will fall into place.

Let $\mathscr{E}^\bullet$ denote the canonical 2-term complex $\mathscr{O}_X \to \mathscr{O}_X(D)$ concentrated in degrees $-1$ and 0, so via the contravariance of $\mathscr{H}om$ in the first variable there is a canonical identification in the derived category

$$\eta : \mathscr{E}^\bullet \otimes \mathscr{F}^\bullet \xleftarrow{\simeq} \mathbf{R}\mathscr{H}om^\bullet_X(\mathscr{O}_D[-1], \mathscr{F}^\bullet) = j_*j^\flat\mathscr{F}^\bullet[1]$$

for any complex $\mathscr{F}^\bullet$ of $\mathscr{O}_X$-modules. When $\mathscr{F}^\bullet = \mathscr{F}[0]$, this recovers on $\mathrm{H}^0$'s exactly the classical method of computing an $\mathscr{E}xt^1$ via a Koszul resolution (up to universal sign). Thus, we get a universally sign-commutative diagram in $\mathbf{D}(X)$:

$$\begin{array}{ccccccc}
\omega_f[n-1] & \longrightarrow & \omega_f(D)[n-1] & \longrightarrow & \mathscr{E}^\bullet \otimes \omega_f[n-1] & \longrightarrow & \omega_f[n] \\
\Vert & & \Vert & & \simeq \big\uparrow \eta & & \\
 & & & & (j_*j^\flat \omega_f[n-1])[1] & & \\
 & & & & \Vert & & \\
\omega_f[n-1] & \longrightarrow & \omega_f(D)[n-1] & \longrightarrow & j_*\omega_g[n-1] & &
\end{array}$$

in which the top row is the mapping cone complex of the bottom row. In particular, going up the right column and then to the term $\omega_f[n]$ at the end of the top row computes $\delta$ up to a universal sign. Since the left column of (1.9) also involves passing through the isomorphism $j_*\omega_g[n-1] \simeq j_*j^\flat\omega_f[n]$ and using the $j^\Delta$-acyclic resolution (1.7), up to universal signs we can identify (1.9) with the outside edge of the diagram

$$\begin{array}{ccccc}
j_*j^\flat(\omega_f[n]) & \xrightarrow{\ \simeq\ } & \mathscr{E}^\bullet \otimes \omega_f[n-1] & \longrightarrow & \omega_f[n] \\
\big\downarrow & & & & \big\downarrow \\
j_*j^\flat \mathscr{H}om^\bullet_X(f^*\mathscr{I}, f^\Delta\mathscr{I}) & \xrightarrow[\simeq]{} & \mathscr{E}^\bullet \otimes \mathscr{H}om^\bullet_X(f^*\mathscr{I}, f^\Delta\mathscr{I})[-1] & \longrightarrow & \mathscr{H}om^\bullet_X(f^*\mathscr{I}, f^\Delta\mathscr{I}) \\
\Vert & & & \nearrow & \\
\mathscr{H}om^\bullet_X(f^*\mathscr{I}, j_*j^\Delta f^\Delta\mathscr{I}) & & & &
\end{array}$$

The commutativity of the top part of this diagram follows from functoriality and the universal sign-commutativity of the bottom part is obvious. This completes the proof.

■

## 2. Trace maps and coboundaries: analytic case

On the analytic side we will be able to use cohomology with compact supports to localize our problems (provided we set things up in adequate generality), and in this way will be able to reduce some questions to the case of an open ball, where explicit computations can be carried out. Let $\mathbf{C}$ denote a fixed algebraic closure of $\mathbf{R}$, equipped with its unique structure of topological field over $\mathbf{R}$ (the only structure which is relevant for the basic definitions in complex analysis). Let $X$ be a paracompact Hausdorff manifold over $\mathbf{C}$ of pure dimension $n > 0$ and let $j : Y \hookrightarrow X$ be a complex submanifold of pure dimension $n-1$. Note that we do not make any connectedness assumptions, for either $X$ or $Y$. This ensures that we can use Bertini's theorem in subsequent applications to bring ourselves down to dimension 0 without worry. Let $\omega_X = \Omega^n_X$ and $\omega_Y = \Omega^{n-1}_Y$ denote the sheaves of top degree holomorphic differential forms on $X$ and $Y$ respectively, and let the $\mathscr{O}_Y$-module $\omega_{Y/X}$ denote the normal sheaf for $Y$ in $X$ (i.e., the dual of the invertible $\mathscr{O}_Y$-module $\mathscr{I}_Y/\mathscr{I}_Y^2$, where $\mathscr{I}_Y = \ker(\mathscr{O}_X \twoheadrightarrow j_*\mathscr{O}_Y)$). For any local generator $t$ of $\mathscr{I}_Y$, we let $t^\vee$ denote the corresponding local dual basis for $\omega_{Y/X}$ over $\mathscr{O}_Y$. There is a canonical isomorphism

$$\omega_{Y/X} \otimes j^*\omega_X \simeq \omega_Y \tag{2.1}$$

locally determined by

$$t^\vee \otimes j^*(\mathrm{d}t \wedge \eta) \mapsto \eta|_Y,$$

where $t$ is a local equation for $Y$ in $X$.

By applying $\omega_X \otimes_{\mathscr{O}_X} (\cdot)$ to the canonical exact sequence

$$0 \to \mathscr{O}_X \to \mathscr{O}_X(Y) \to \omega_{Y/X} \to 0$$

with the right map given locally by $1/t \mapsto t^\vee$, we therefore get an exact sequence

$$0 \to \omega_X \to \omega_X(Y) \to \omega_Y \to 0$$

from which arises a $\mathbf{C}$-linear coboundary map

$$\delta : \mathrm{H}^{n-1}_{\mathrm{c}}(Y, \omega_Y) \to \mathrm{H}^n_{\mathrm{c}}(X, \omega_X). \tag{2.2}$$

In order to avoid sign confusion, let us explicate a couple of things related to integration. For a complex manifold $Z$ with local holomorphic coordinates $z_1, \dots, z_m$ and a choice of $i = \sqrt{-1} \in \mathbf{C}$, the orientation $1 \wedge i > 0$ of the underlying $\mathbf{R}$-manifold of $\mathbf{C}$ induces as usual an orientation on the $\mathbf{R}$-manifold underlying $Z$ (called the *$i$-orientation* of $Z$), independent of the choice of local holomorphic coordinates. This corresponds of course to the "positivity" of the smooth "$\mathbf{R}$-valued" differential form

$$(i\mathrm{d}z_1 \wedge \mathrm{d}\overline{z}_1) \wedge \cdots \wedge (i\mathrm{d}z_m \wedge \mathrm{d}\overline{z}_m) \tag{2.3}$$

on the underlying $\mathbf{R}$-manifold, and (2.3) depends on the choice of $i$ up to a sign of $(-1)^m$.

For such paracompact Hausdorff $Z$ with pure dimension $m$, the resulting integration map

$$\int_Z : \mathrm{H}^{2m}_{\mathrm{c}}(Z, \underline{\mathbf{C}}) \to \mathbf{C}$$

therefore depends on the choice of $i$ up to a sign of $(-1)^m$, so the map

$$\mathrm{Tr}_Z \stackrel{\mathrm{def}}{=} \frac{1}{(2\pi i)^m} \int_Z : \mathrm{H}^{2m}_{\mathrm{c}}(Z, \underline{\mathbf{C}}) \to \mathbf{C}$$

is independent of the choice of $i$. Without such independence, this could not possibly be compatible with any purely algebro-geometric construction over $\mathbf{C}$. Likewise, since the holomorphic deRham complex gives a resolution of the constant sheaf $\underline{\mathbf{C}}$, the spectral sequence for hypercohomology gives rise to a Hodge-to-deRham spectral sequence

$$\mathrm{H}^p_{\mathrm{c}}(Z, \Omega^q_Z) \Rightarrow \mathrm{H}^{p+q}_{\mathrm{c}}(Z, \underline{\mathbf{C}})$$

that may be computed using the classical Hodge-deRham double complex of fine sheaves $\mathscr{A}^{p,q}_Z$ of type-$(p,q)$ smooth $\mathbf{C}$-valued differential forms on the underlying $\mathbf{R}$-manifold. In particular, we get a canonical isomorphism

$$\xi_Z : \mathrm{H}^m_{\mathrm{c}}(Z, \Omega^m_Z) \simeq \mathrm{H}^{2m}_{\mathrm{c}}(Z, \underline{\mathbf{C}}).$$

Now consider the diagram

$$\begin{array}{ccc}
\mathrm{H}^{n-1}_{\mathrm{c}}(Y, \omega_Y) & \xrightarrow{\delta} & \mathrm{H}^n_{\mathrm{c}}(X, \omega_X) \\
\simeq \big\downarrow \xi_Y & & \xi_X \big\downarrow \simeq \\
\mathrm{H}^{2(n-1)}_{\mathrm{c}}(Y, \underline{\mathbf{C}}) & & \mathrm{H}^{2n}_{\mathrm{c}}(X, \underline{\mathbf{C}}) \\
\mathrm{Tr}_Y \big\downarrow & & \big\downarrow \mathrm{Tr}_X \\
\mathbf{C} & \xrightarrow[\mathrm{id}]{} & \mathbf{C}
\end{array} \tag{2.4}$$

with top row as in (2.2). The first thing we wish to prove is:

**Theorem 2.1.** *The diagram* (2.4) *commutes up to a sign of* $(-1)^n$.

*Proof.* Although the main case of interest is perhaps when $X$ and $Y$ are compact, by working more generally with compactly supported cohomology we shall be able to reduce ourselves to a local calculation on the unit ball in $\mathbf{C}^n$. In order to carry out the localization, let $U \subseteq X$ be open and let $V = j^{-1}(U) = U \cap Y$ be the open overlap with $Y$. Recall that for any sheaf $\mathscr{F}$ on $X$ there is a canonical map

$$\mathrm{H}^\bullet_{\mathrm{c}}(U, \mathscr{F}|_U) \to \mathrm{H}^\bullet_{\mathrm{c}}(X, \mathscr{F}) \tag{2.5}$$

uniquely determined by universal $\delta$-functoriality and the evident specification in degree 0. More specifically, if $\mathscr{F} \to \mathscr{I}^\bullet$ is a fine resolution (e.g., the $\overline{\partial}$-resolution $\Omega_X^p \to \mathscr{A}_X^{p,\bullet}$) then so is $\mathscr{F}|_U \to \mathscr{I}^\bullet|_U$ and

$$\Gamma_{\mathrm{c}}(U, \mathscr{I}^\bullet|_U) \to \Gamma_{\mathrm{c}}(X, \mathscr{I}^\bullet)$$

induces the canonical $\delta$-functorial map (2.5) on compactly supported cohomology. Using this explication, as well as the functorial construction of the Hodge-to-deRham spectral sequence in terms of the double complex $(\mathscr{A}^{\bullet\bullet}, \partial, \overline{\partial})$ of fine sheaves, we see that the diagram (2.4) for the pair $(X, Y)$ is compatible with the analogous diagram for the pair $(U, V)$.

Thus, to prove the theorem for $(X, Y)$ it is enough to prove it for a set of pairs $(U_\alpha, V_\alpha)$ such that the maps

$$\mathrm{H}_{\mathrm{c}}^{2(n-1)}(V_\alpha, \underline{\mathbf{C}}) \to \mathrm{H}_{\mathrm{c}}^{2(n-1)}(Y, \underline{\mathbf{C}})$$

have images spanning $\mathrm{H}_{\mathrm{c}}^{2(n-1)}(Y, \underline{\mathbf{C}})$ as a $\mathbf{C}$-vector space. Moreover, we can replace $X$ with any open neighborhood of $Y$. Since we can take $Y$ to be connected (as cohomology with compact supports take disjoint unions to direct sums), we then have that

$$\mathrm{H}_{\mathrm{c}}^{2(n-1)}(V, \underline{\mathbf{C}}) \simeq \mathrm{H}_{\mathrm{c}}^{2(n-1)}(Y, \underline{\mathbf{C}})$$

for any small open ball $V$ in $Y$. In this way, we see that we may reduce ourselves to the case in which $X$ is the "ball" of points $z \in \mathbf{C}^n$ for which $|z_j| < 1$ for all $j$, and $Y$ is the $(n-1)$-dimensional "ball" in here cut out by the condition $z_n = 0$.

We now carry out the analysis by making an explicit exact sequence of fine resolutions over the short exact sequence

$$0 \to \omega_X \to \omega_X(Y) \to \omega_Y \to 0$$

as defined above. We certainly have the $\mathscr{O}_X$-linear $\overline{\partial}$-resolutions

$$\omega_X \to \mathscr{A}_X^{n,\bullet}$$

and

$$\omega_X(Y) \to \mathscr{A}_X^{n,\bullet}(Y) \stackrel{\mathrm{def}}{=} \mathscr{A}_X^{n,\bullet} \otimes_{\mathscr{O}_X} \mathscr{O}_X(Y).$$

Since $z_n$ is nowhere a zero-divisor on $\mathscr{A}_X^{0,0}$ and all $\mathscr{A}_X^{p,q}$ are locally free over $\mathscr{A}_X^{0,0}$, we see that

$$\mathrm{Tor}^i_{\mathscr{O}_{X,x}}(\mathscr{O}_{X,x}/z_n, \mathscr{A}_{X,x}^{p,q}) = 0$$

for all $x \in X$, $i \geq 1$, and $p, q \geq 0$. Thus,

$$j^*\omega_X \to j^*\mathscr{A}_X^{n,\bullet} = \mathscr{A}_X^{n,\bullet}/z_n$$

is a resolution, visibly by fine sheaves (of $\mathscr{A}_X^{0,0}/z_n$-modules).

Thus, we get a natural diagram with exact rows and resolutions in the columns:

(2.6)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{A}_X^{n,\bullet} & \longrightarrow & \mathscr{A}_X^{n,\bullet}(Y) & \longrightarrow & \omega_{Y/X} \otimes j^*\mathscr{A}_X^{n,\bullet} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \omega_X & \longrightarrow & \omega_X(Y) & \longrightarrow & \omega_{Y/X} \otimes j^*\omega_X & \longrightarrow & 0 \end{array}$$

In order to connect up this diagram with (2.1), we must link up the right column of (2.6) and the fine resolution $\omega_Y \to \mathscr{A}_Y^{n-1,\bullet}$, compatibly with (2.1). Let $\pi : X \to Y$ be the holomorphic projection

$$(z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_{n-1}, 0),$$

so $\pi \circ j$ is the identity on $Y$. Consider the map of sheaves

$$\mathscr{A}_Y^{n-1,q} \to \omega_{Y/X} \otimes j^*\mathscr{A}_X^{n,q}$$

defined on local sections by

(2.7)
$$\eta \mapsto z_n^\vee \otimes j^*(\mathrm{d}z_n \wedge \pi^*\eta).$$

Since $\overline{\partial}(\mathrm{d}z_n \wedge \pi^*\eta) = -\mathrm{d}z_n \wedge \pi^*\overline{\partial}\eta$, in order to define a *commutative* diagram of fine resolutions

$$\begin{array}{ccc} \mathscr{A}_Y^{n-1,\bullet} & \longrightarrow & \omega_{Y/X} \otimes j^*\mathscr{A}_X^{n,\bullet} \\ \uparrow & & \uparrow \\ \omega_Y & \xrightarrow[\simeq]{} & \omega_{Y/X} \otimes j^*\omega_X \end{array} \tag{2.8}$$

we replace (2.7) with

$$\eta \mapsto \varepsilon_q z_n^\vee \otimes j^*(\mathrm{d}z_n \wedge \pi^*\eta) \tag{2.9}$$

in degree $q$, with $\varepsilon_{q+1} = -\varepsilon_q$ and $\varepsilon_0 = 1$ (to respect augmentation), so $\varepsilon_q = (-1)^q$ for all $q$. By the general theory of fine resolutions on paracompact Hausdorff spaces, we conclude that the maps between the two columns in (2.8) induce isomorphisms on $\Gamma_c(Y, \cdot)$-cohomology in all degrees.

Now we are ready to chase through the snake lemma using (2.6) and (2.8) to explicate going around (2.4) from the middle of the left column, across the top, and down to the middle of the right column. Pick a compactly supported $(n-1, n-1)$-form $\eta$ on $Y$ representing a class in $\mathrm{H}_c^{2(n-1)}(Y, \underline{\mathbf{C}})$, so (2.9) leads us to consider

$$(-1)^{n-1} z_n^\vee \otimes j^*(\mathrm{d}z_n \wedge \pi^*\eta), \tag{2.10}$$

a compactly supported global section of $\omega_{Y/X} \otimes j^*\mathscr{A}_X^{n,n-1}$ (note the support is still in $Y$, whence its compactness). The general nonsense concerning fine sheaves ensures that (2.10) lifts to a *compactly supported* global section $\theta$ of $\mathscr{A}_X^{n,n-1}(Y)$ via the $(n-1)$th row of (2.6), and moreover $\overline{\partial}\theta \in \Gamma_c(\mathscr{A}_X^{n,n}(Y))$ must actually be inside of $\Gamma_c(\mathscr{A}_X^{n,n})$. Let $\omega \in \Gamma_c(\mathscr{A}_X^{n,n})$ denote this element. This represents a cohomology class in $\mathrm{H}_c^{2n}(X, \underline{\mathbf{C}})$ which is the image of our initial class of $\eta$ upon going around (2.4) from the middle left, up across the top, and down to the middle right. Thus, the desired $(-1)^n$-commutativity of (2.4) says exactly

$$\frac{1}{(2\pi i)^n} \int_X \omega = \frac{(-1)^n}{(2\pi i)^{n-1}} \int_Y \eta.$$

Recalling the sign of $(-1)^{n-1}$ that entered into the above "construction" of $\omega$ from $\eta$, we could alternatively have chosen $\theta_0 \in \Gamma_c(\mathscr{A}_X^{n,n-1}(Y))$ with

$$\theta_0 \mapsto z_n^\vee \otimes j^*(\mathrm{d}z_n \wedge \pi^*\eta) \in \Gamma_c(\omega_{Y/X} \otimes j^*\mathscr{A}_X^{n,n-1})$$

under (2.7), and then for $\omega_0 = \overline{\partial}\theta_0 \in \Gamma_c(\mathscr{A}_X^{n,n}) \subseteq \Gamma_c(\mathscr{A}_X^{n,n}(Y))$ we want

$$\frac{1}{(2\pi i)^n} \int_X \omega_0 = \frac{-1}{(2\pi i)^{n-1}} \int_Y \eta. \tag{2.11}$$

More specifically, in order to avoid "denominators" (i.e., simple poles along $Y$) in the construction, we choose $\theta_1 \in \Gamma_c(\mathscr{A}_X^{n,n-1})$ with

$$\theta_1 \mapsto j^*(\mathrm{d}z_n \wedge \pi^*\eta) \tag{2.12}$$

under $\Gamma_c(\mathscr{A}_X^{n,n-1}) \to \Gamma_c(j^*\mathscr{A}_X^{n,n-1})$ and then by general theory we *must* have $\overline{\partial}\theta_1 = z_n\omega_0$ for some $\omega_0 \in \Gamma_c(\mathscr{A}_X^{n,n})$. We then wish to prove (2.11) for this $\omega_0$. We shall prove the more precise statement

$$\eta = \frac{-1}{2\pi i} \int_{|z_n|<1} \omega_0$$

(where we just "integrate out $z_n$"), and so by Fubini's Theorem (2.11) will follow.

Now it is time for the local computation. We may uniquely write

$$\eta = g\,\mathrm{d}\overline{z}_1 \wedge \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}\overline{z}_{n-1} \wedge \mathrm{d}z_{n-1} \tag{2.13}$$

for a compactly supported $g$ and

$$\theta_1 = \sum_{k=1}^n f_k\,\mathrm{d}z_k \wedge \mathrm{d}\overline{z}_1 \wedge \mathrm{d}z_1 \wedge \cdots \wedge \widehat{\mathrm{d}\overline{z}_k \wedge \mathrm{d}z_k} \wedge \cdots \wedge \mathrm{d}\overline{z}_n \wedge \mathrm{d}z_n$$

for compactly supported $f_j$'s, where the $\widehat{\cdot}$ means we delete that term. But we selected $\theta_1$ so that its image in $j^*\mathscr{A}_X^{n,n-1}$ is

$$j^*(\mathrm{d}z_n \wedge \pi^*\eta) = j^*(g\,\mathrm{d}z_n \wedge \mathrm{d}\overline{z}_1 \wedge \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}\overline{z}_{n-1} \wedge \mathrm{d}z_{n-1}),$$

which involves no terms of type

$$\mathrm{d}z_k \wedge \mathrm{d}\overline{z}_1 \wedge \mathrm{d}z_1 \wedge \cdots \wedge \widehat{\mathrm{d}\overline{z}_k \wedge \mathrm{d}z_k} \wedge \cdots \wedge \mathrm{d}\overline{z}_{n-1} \wedge \mathrm{d}z_{n-1}$$

for $k < n$. Thus, by consideration of bases of sheaves of modules, it follows that for $1 \leq k < n$ all terms

$$j^*(f_k\,\mathrm{d}z_k \wedge d\overline{z}_1 \wedge \mathrm{d}z_1 \wedge \cdots \wedge \widehat{\mathrm{d}\overline{z}_k \wedge \mathrm{d}z_k} \wedge \cdots \wedge \mathrm{d}\overline{z}_n \wedge \mathrm{d}z_n)$$

vanish. Hence, since the only thing that matters for the choice of $\theta_1$ is its lifting property (2.12) with respect to $j^*(\mathrm{d}z_n \wedge \pi^*\eta)$, we may suppose $f_k = 0$ for all $k < n$. That is, we may suppose

$$\theta_1 = f\,\mathrm{d}z_n \wedge \mathrm{d}\overline{z}_1 \wedge \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}\overline{z}_{n-1} \wedge \mathrm{d}z_{n-1} \tag{2.14}$$

for a compactly supported $f$.

Using (2.13), (2.14), and the property $\pi \circ j = \mathrm{id}_Y$, the condition (2.12) says exactly that

$$f(z_1, \ldots, z_n) = g(z_1, \ldots, z_{n-1}) + z_n\varphi \tag{2.15}$$

for some $\varphi \in \Gamma(X, \mathscr{A}_X^{0,0})$. Note that $g$ as a function of $(z_1, \ldots, z_n)$ is *not* compactly supported, so $\varphi$ generally won't have compact support either. In any case, we compute from (2.15) that

$$\frac{\partial f}{\partial \overline{z}_n} = z_n \frac{\partial \varphi}{\partial \overline{z}_n}, \tag{2.16}$$

so

$$\begin{aligned} g(z_1, \ldots, z_{n-1}) &= f(z_1, \ldots, z_{n-1}, 0) \\ &= \frac{1}{2\pi i} \int_{|z_n|<1} \frac{\partial f/\partial \overline{z}_n}{z_n - 0}\,\mathrm{d}z_n \wedge \mathrm{d}\overline{z}_n \\ &= \frac{1}{2\pi i} \int_{|z_n|<1} \frac{\partial \varphi}{\partial \overline{z}_n}\,\mathrm{d}z_n \wedge \mathrm{d}\overline{z}_n \\ &= \frac{1}{2\pi i} \int_{|z_n|<1} -\frac{\partial \varphi}{\partial \overline{z}_n}\,\mathrm{d}\overline{z}_n \wedge \mathrm{d}z_n, \end{aligned}$$

where the second equality uses the fact that $f$ has *compact* support (thereby killing the boundary integral in the generalized Cauchy integral formula).

Recalling that $\eta = g\,\mathrm{d}\overline{z}_1 \wedge \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}\overline{z}_{n-1} \wedge \mathrm{d}z_{n-1}$, we conclude from this explicit "computation" of $g$ that

$$\eta = \frac{1}{2\pi i} \int_{|z_n|<1} -\frac{\partial \varphi}{\partial \overline{z}_n}\,\mathrm{d}\overline{z}_n \wedge \mathrm{d}z_n \wedge \mathrm{d}\overline{z}_1 \wedge \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}\overline{z}_{n-1} \wedge \mathrm{d}z_{n-1}.$$

In order to conclude that $\eta = -\frac{1}{2\pi i}\int_{|z_n|<1}\omega_0$, it suffices to show

$$\omega_0 = \frac{\partial \varphi}{\partial \overline{z}_n}\,\mathrm{d}\overline{z}_n \wedge \mathrm{d}z_n \wedge \mathrm{d}\overline{z}_1 \wedge \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}\overline{z}_{n-1} \wedge \mathrm{d}z_{n-1}.$$

But by (2.14), (2.16), and the definition of $\omega_0$ we have

$$\begin{aligned} z_n\omega_0 &= \overline{\partial}\theta_1 \\ &= \overline{\partial}(f\,\mathrm{d}z_n \wedge \mathrm{d}\overline{z}_1 \wedge \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}\overline{z}_{n-1} \wedge \mathrm{d}z_{n-1}) \\ &= \frac{\partial f}{\partial \overline{z}_n}\,\mathrm{d}\overline{z}_n \wedge \mathrm{d}z_n \wedge \mathrm{d}\overline{z}_1 \wedge \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}\overline{z}_{n-1} \wedge \mathrm{d}z_{n-1} \\ &= z_n \frac{\partial \varphi}{\partial \overline{z}_n}\,\mathrm{d}\overline{z}_n \wedge \mathrm{d}z_n \wedge \mathrm{d}\overline{z}_1 \wedge \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}\overline{z}_{n-1} \wedge \mathrm{d}z_{n-1} \end{aligned}$$

Now cancelling $z_n$ completes the proof of the Theorem.

■

## 3. Comparison of traces

Let $X$ denote a smooth proper scheme over $\mathrm{Spec}(\mathbf{C})$, with pure dimension $n \geq 0$. Using the isomorphism $\omega_X^{\mathrm{an}} \simeq \omega_{X^{\mathrm{an}}}$, there is a natural diagram

$$\begin{array}{ccc} \mathrm{H}^n(X, \omega_X) & \longrightarrow & \mathrm{H}^n(X^{\mathrm{an}}, \omega_{X^{\mathrm{an}}}) \\ {\scriptstyle \mathrm{Tr}_X}\downarrow & & \downarrow\simeq \\ \mathbf{C} & \xleftarrow[\mathrm{Tr}_{X^{\mathrm{an}}}]{} & \mathrm{H}^{2n}(X^{\mathrm{an}}, \underline{\mathbf{C}}) \end{array} \tag{3.1}$$

Although Grothendieck's generalization of GAGA in Exp. XIII of SGA1 implies that the top horizontal map is an isomorphism, we do not need this fact. We wish to investigate the nature of the commutativity or otherwise of (3.1).

**Theorem 3.1.** *The diagram* (3.1) *commutes up to a sign of* $(-1)^n$.

*Proof.* We first will use pure thought to prove the theorem up to a universal undetermined constant depending only on $n$. This constant will then be computed by considering a *single* example in dimension $n$ (namely, $\mathbf{P}^n$). Choose a closed point $x \in X$. We also view $x$ as a (closed) point of $X^{\mathrm{an}}$. By universal $\delta$-functor considerations, there is a commutative diagram of $\delta$-functorial morphisms

$$\begin{array}{ccc} \mathrm{H}^\bullet_{\{x\}}(X, \cdot) & \longrightarrow & \mathrm{H}^\bullet(X, \cdot) \\ \downarrow & & \downarrow \\ \mathrm{H}^\bullet_{\{x\}}(X^{\mathrm{an}}, (\cdot)^{\mathrm{an}}) & \longrightarrow & \mathrm{H}^\bullet(X^{\mathrm{an}}, (\cdot)^{\mathrm{an}}) \end{array} \tag{3.2}$$

The arguments near [C2, p. 6, line 14] show that the composite map

$$\mathrm{H}^\bullet_{\{x\}}(X, \omega_X) \to \mathrm{H}^\bullet(X, \omega_X) \xrightarrow{\mathrm{Tr}_X} \mathbf{C}$$

is equal to $(-1)^n \mathrm{Tr}_{X,\{x\}}$, where $\mathrm{Tr}_{X,\{x\}}$ is the "local trace" defined by [C1, (A.2.16)]. If $z_1, \ldots, z_n$ are sections of $\mathscr{O}_X$ on a neighborhood of $x$ which induce a regular system of parameters in the local ring at $x$, then the recipe in [C1, pp. 253–4] defines a "fraction"

$$\eta = \frac{\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n}{z_1 \cdots z_n} \in \mathrm{H}^n_{\{x\}}(X, \omega_X)$$

and [C1, Lemma A.2.1] implies that $\mathrm{Tr}_{X,x}(\eta) = 1$. This computation makes essential use of Grothendieck's general theory of the residue symbol.

Since the vector space $\mathrm{H}^n(X, \omega_X)$ is 1-dimensional over $\mathbf{C}$ and there is a canonical isomorphism $\omega_X^{\mathrm{an}} \simeq \omega_{X^{\mathrm{an}}}$, it makes sense to let $\eta^{\mathrm{an}}$ denote the image of $\eta$ under the left vertical arrow in (3.2) and thus the theorem for $X$ is *equivalent* to the assertion that the composite map

$$\mathrm{H}^\bullet_{\{x\}}(X^{\mathrm{an}}, \omega_{X^{\mathrm{an}}}) \to \mathrm{H}^\bullet(X^{\mathrm{an}}, \omega_{X^{\mathrm{an}}}) \xrightarrow{\mathrm{Tr}_{X^{\mathrm{an}}}} \mathbf{C}$$

sends $\eta^{\mathrm{an}}$ to 1.

If $U$ denotes a Zariski open around $x$ on which $z_1, \ldots, z_n$ are defined, then the analytic map

$$\varphi \stackrel{\mathrm{def}}{=} (z_1, \ldots, z_n) : U^{\mathrm{an}} \to \mathbf{C}^n$$

is étale at $x$. Thus, for a sufficiently small open ball $\Delta_{r_x}$ of some radius $r_x > 0$ around the origin in $\mathbf{C}^n$ we may identify $\Delta_{r_x}$ with an open neighborhood of $x$ in $U^{\mathrm{an}}$ via $\varphi$. We wish to use this to prove that the theorem for $X$ is logically equivalent to a computation in local cohomology on the open unit ball in $\mathbf{C}^n$. By universal $\delta$-functor arguments, if $i : V \to M$ is an open immersion between paracompact Hausdorff ringed spaces and $x \in V$ is a point, then we have a commutative $\delta$-functorial diagram

$$\begin{array}{ccc} \mathrm{H}^\bullet_{\{x\}}(V, (\cdot)|_V) & \longrightarrow & \mathrm{H}^\bullet_{\mathrm{c}}(V, (\cdot)|_V) \\ \downarrow & & \downarrow \\ \mathrm{H}^\bullet_{\{x\}}(M, \cdot) & \longrightarrow & \mathrm{H}^\bullet_{\mathrm{c}}(M, \cdot) \end{array} \tag{3.3}$$

which is uniquely determined by the natural maps in degree 0. Here, the "variable" is an arbitrary $\mathscr{O}_M$-module. Since the right vertical map admits a concrete description on the level of fine resolutions, we see with the help of deRham resolutions that if $i : V \to M$ is an open immersion of paracompact Hausdorff complex manifolds of pure dimension $n$, then composition with the map

$$\mathrm{H}^n_{\mathrm{c}}(V, \omega_V) \to \mathrm{H}^n_{\mathrm{c}}(M, \omega_M)$$

carries $\mathrm{Tr}_M$ over to $\mathrm{Tr}_V$.

Applying these considerations to the open immersion $\Delta_{r_x} \hookrightarrow X^{\mathrm{an}}$, we conclude that the theorem for $X$ is *equivalent* to the assertion that the composite

$$\mathrm{H}^n_{\{0\}}(\Delta_{r_x}, \omega_{\Delta_{r_x}}) \to \mathrm{H}^n_{\mathrm{c}}(\Delta_{r_x}, \omega_{\Delta_{r_x}}) \stackrel{\mathrm{Tr}_{\Delta_{r_x}}}{\longrightarrow} \mathbf{C} \tag{3.4}$$

sends the "fraction"

$$\frac{\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n}{z_1 \cdots z_n} \tag{3.5}$$

to 1, where $z_1, \ldots, z_n$ are the standard ordered coordinates on $\Delta_{r_x}$. Here we have implicitly used the fact that the scheme-theoretic construction in [C1, p. 253] also makes sense in the complex analytic case, where [C1, (A.2.18)] is viewed as simply a natural transformation of $\delta$-functors on sheaves of modules (generally not an isomorphism) and the Koszul complexes in this construction are computed with module stalks over local rings at points (as opposed to modules of sections over an open affine as in the scheme case with quasi-coherent sheaves). In order to see that this Koszul construction really $\delta$-functorially computes the module Ext's in the analytic case, just as in the scheme case, one uses an erasable $\delta$-functor argument. This analytic "fraction" construction is visibly compatible with analytificiation from the algebraic case.

Observe that the holomorphic functions $y_j = z_j / r_x$ are also local equations cutting out the reduced point 0, so the "fraction"

$$\frac{\mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_n}{y_1 \cdots y_n} \tag{3.6}$$

makes sense and coincides with (3.5) in local cohomology at the origin. The reason that (3.5) coincides with (3.6) is not because of the suggestive fraction notation (which is purely symbolic) but rather because

$$\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n = r_x^n \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_n$$

in the stalk $(\omega_{\Delta_{r_x}})_0$ at the origin and the natural isomorphism of Koszul complexes

$$K^\bullet(\mathbf{z}, \mathscr{F}_0) \simeq K^\bullet(\mathbf{y}, \mathscr{F}_0)$$

lifting the identity in degree 0 is given by multiplication by $r_x^{-j}$ in degree $j$ (and hence multiplication by $r_x^{-n}$ in degree $n$).

Since (3.4) is intrinsically defined for any pointed paracompact Hausdorff manifold and is visibly functorial with respect to isomorphisms, the isomorphism $\Delta_{r_x} \simeq \Delta \stackrel{\mathrm{def}}{=} \Delta_1$ defined by

$$(z_1, \ldots, z_n) \mapsto (z_1/r_x, \ldots, z_n/r_x)$$

identifies our local cohomology claim on $\Delta_{r_x}$ concerning (3.4) and (3.5) with the analogous assertion on the open unit ball $\Delta$ in $\mathbf{C}^n$. Note that this equivalence rests on the equality of (3.5) and (3.6) in local cohomology on $\Delta_{r_x}$. Summarizing the conclusion of this general nonsense, the $(-1)^n$-commutativity of (3.1) for a fixed $X$ is equivalent to the assertion that the composite

$$\mathrm{H}^n_{\{0\}}(\Delta, \omega_\Delta) \to \mathrm{H}^n_{\mathrm{c}}(\Delta, \omega_\Delta) \xrightarrow{\mathrm{Tr}_\Delta} \mathbf{C} \tag{3.7}$$

sends (3.5) to 1, where $z_1, \dots, z_n$ denote the standard ordered holomorphic coordinates on the open unit ball $\Delta$ in $\mathbf{C}^n$. This latter statement is *independent* of $X$, so knowing the theorem in a single example is sufficient to deduce the general case!

We now give a general induction-on-$n$ proof in the projective case, using hyperplane slices and Theorems 1.1 and 2.1. The case $n = 0$ is clear, so we may (and do) assume $n \geq 1$. Since we only need to consider the single case $X = \mathbf{P}^n$, and Theorem 1.1 for a hyperplane $\mathbf{P}^{n-1}$ embedded in $\mathbf{P}^n$ is established by direct computation as the first step in the *proof* of Theorem 1.1, for the argument that follows we only need this very special (and down-to-earth) projective space case of Theorem 1.1. With these preliminary remarks made, we now carry out the induction. Since $n > 0$, we may use Bertini's theorem to find a smooth hyperplane section $j : Y \hookrightarrow X$ (so $Y$ is of pure dimension $n - 1$, typically disconnected if $n = 1$). We saw in Theorem 2.1 that the analytic coboundary map

$$\delta^{\mathrm{an}} : \mathrm{H}^{n-1}(Y^{\mathrm{an}}, \omega_{Y^{\mathrm{an}}}) \to \mathrm{H}^n(X^{\mathrm{an}}, \omega_{X^{\mathrm{an}}})$$

satisfies

$$\mathrm{Tr}_{X^{\mathrm{an}}} \circ \delta^{\mathrm{an}} = (-1)^n \mathrm{Tr}_{Y^{\mathrm{an}}}. \tag{3.8}$$

But by Theorem 1.1 (which we really only need in the easy case of $\mathbf{P}^{n-1} \hookrightarrow \mathbf{P}^n$), the (surjective!) algebraic coboundary map

$$\delta^{\mathrm{alg}} : \mathrm{H}^{n-1}(Y, \omega_Y) \twoheadrightarrow \mathrm{H}^n(X, \omega_X)$$

satisfies

$$\mathrm{Tr}_X \circ \delta^{\mathrm{alg}} = (-1)^{n-1} \mathrm{Tr}_Y \tag{3.9}$$

so the inductive hypothesis for $Y$, along with the surjectivity of $\delta^{\mathrm{alg}}$, permits us to conclude that (3.1) commutes up to a sign of

$$(-1)^{n-1}(-1)^n(-1)^{n-1} = (-1)^n$$

as desired. In more down-to-earth terms, (3.8) and (3.9) imply that each time we drop the dimension by 1, the analytic and algebraic theories have a discrepency of $(-1)^n(-1)^{n-1} = -1$, and so since we have compatibility upon reaching dimension 0 we conclude that the total discrepancy in dimension $n$ is $(-1)^n$, as desired.

■

Now that we have taken care of the general proper case over Spec($\mathbf{C}$), it is natural to ask about the relative case. This will be almost a trivial consequence of what has gone before, but a little thought will be needed to get around nilpotent issues. First let us formulate what is to be shown. In Grothendieck's theory one has a good notion of relative trace for proper smooth maps, compatible with arbitrary base change. Meanwhile, the relative analytic trace goes as follows. If $f : X \to S$ is a proper smooth map between complex analytic spaces and $f$ has fibers of pure dimension $n$, the relative holomorphic deRham complex gives a resolution of $f^{-1}\mathscr{O}_S$, and from this one gets a canonical $\mathscr{O}_S$-linear isomorphism

$$\mathrm{R}^n f_*(\Omega^n_{X/S}) \simeq \mathrm{R}^{2n} f_*(f^{-1}\mathscr{O}_S) \simeq \mathrm{R}^{2n} f_*(\underline{\mathbf{C}}) \otimes_{\underline{\mathbf{C}}} \mathscr{O}_S$$

compatible with base change and recovering the classical isomorphism on fibers. Moreover, by keeping careful track of orientations there is a unique $\underline{\mathbf{C}}$-linear isomorphism

$$\mathrm{R}^{2n} f_*(\underline{\mathbf{C}}) \simeq \underline{\mathbf{C}} \tag{3.10}$$

which is the classical analytic trace on fibers (existence of (3.10) with the specified fiber properties is the point here, of course). Putting these together, we get an $\mathscr{O}_S$-linear relative analytic trace map

$$\mathrm{Tr}_{X/S} : \mathrm{R}^n f_*(\Omega^n_{X/S}) \to \mathscr{O}_S$$

which respects base change and induces the usual analytic trace on fibers. It makes now sense to state:

**Theorem 3.2.** *Let $f$ be a proper smooth morphism between locally finite type* $\mathbf{C}$*-schemes. Then*

$$\mathrm{Tr}_{f^{\mathrm{an}}} = (-1)^n (\mathrm{Tr}_f)^{\mathrm{an}}.$$

*Proof.* By analytic/algebraic cohomology and base change, one immediately reduces to the case of a map $f : X \to S$ where $S = \mathrm{Spec}(A)$ for a finite local $\mathbf{C}$-algebra $A$. We can then also assume $X$ is connected. We have two $A$-linear trace maps

$$\mathrm{H}^n(X, \Omega^n_{X/A}) \to A, \quad \mathrm{H}^n(X^{\mathrm{an}}, \Omega^n_{X^{\mathrm{an}}/A}) \to A$$

which we want to be $(-1)^n$-compatible with respect to the $A$-linear GAGA isomorphism

$$\mathrm{H}^n(X, \Omega^n_{X/A}) \simeq \mathrm{H}^n(X^{\mathrm{an}}, \Omega^n_{X^{\mathrm{an}}/A}).$$

A priori we only know this compatibility modulo the maximal ideal of $A$. To make a comparison more feasible, we shall use the crutch of a choice of section $s \in X(A)$ (which exists since $X$ is $A$-smooth and $A$ is strictly henselian). Since analytification is compatible with formation of $\mathbf{R}\mathscr{H}om$ on coherent sheaves (thanks to GAGA for $\mathscr{E}xt$-sheaves), we have an easy compatibility between the algebraic trace map for the closed immersion $s$ and the analytic trace map for the closed immersion $s^{\mathrm{an}}$ (the latter defined much like the former in terms of formal manipulations with $\mathbf{R}\mathscr{H}om$'s).

In other words, there are canonical compatible maps

$$\mathrm{Tr}_s : A \to \mathrm{H}^n(X, \Omega^n_{X/A}), \quad \mathrm{Tr}_{s^{\mathrm{an}}} : A \to \mathrm{H}^n(X^{\mathrm{an}}, \Omega^n_{X^{\mathrm{an}}/A})$$

which coincide under the GAGA isomorphisms. We first claim these maps are *isomorphisms*. It suffices to check $\mathrm{Tr}_s$ is an isomorphism, but $\mathrm{Tr}_f$ is an isomorphism (thanks to the connectedness hypothesis) and $\mathrm{Tr}_f \circ \mathrm{Tr}_s : A \to A$ is multiplication by a universal sign depending only on $n$. Thus, $\mathrm{Tr}_s$ and $\mathrm{Tr}_{s^{\mathrm{an}}}$ are isomorphisms. It is therefore necessary and sufficient to show

$$\mathrm{Tr}_{f^{\mathrm{an}}} \circ \mathrm{Tr}_{s^{\mathrm{an}}} = (-1)^n \mathrm{Tr}_f \circ \mathrm{Tr}_s$$

as $A$-linear endomorphisms of $A$. But observe that everything works modulo the maximal ideal of $A$; this is where we make essential use of the fact that we have already handled the proper smooth case over $\mathrm{Spec}(\mathbf{C})$. This verification on the closed fiber makes our problem logically equivalent to the following assertion purely on the analytic side: our abstract $A$-linear endomorphism $\mathrm{Tr}_{f^{\mathrm{an}}} \circ \mathrm{Tr}_{s^{\mathrm{an}}}$ of $A$ should be an extension of scalars of a $\mathbf{C}$-linear endomorphism of $\mathbf{C}$. Looking back at the definition of $\mathrm{Tr}_{f^{\mathrm{an}}}$ in terms of a local system, our problem is a special case of the following purely analytic assertion: if $X$ is a proper smooth analytic space of pure relative dimension $n$ over a finite local $\mathbf{C}$-algebra $A$ and if $s \in X(A)$, then the canonical trace map

$$\mathrm{Tr}_s : A \to \mathrm{H}^n(X, \Omega^n_{X/A}) \simeq A \otimes_{\mathbf{C}} \mathrm{H}^{2n}(X, \underline{\mathbf{C}})$$

sends 1 into $\mathrm{H}^{2n}(X, \underline{\mathbf{C}})$. The only sense in which this is more general than the preceding case is that we don't assume $X$ to be "algebraic" (note that for "algebraic" $X$, all analytic sections $s \in X(A)$ are automatically "algebraic" by GAGA).

By using cohomology with compact support and requiring the pure $n$-dimensional $A$-smooth $X$ to be merely paracompact Hausdorff rather than compact Hausdorff, we still have the $A$-linear Hodge-deRham isomorphism for $\mathrm{H}^\bullet_{\mathrm{c}}$'s in top degree and hence we can contemplate the analgous assertion that

$$\mathrm{Tr}_s : A \to \mathrm{H}^n_{\mathrm{c}}(X, \Omega^n_{X/A}) \simeq A \otimes_{\mathbf{C}} \mathrm{H}^{2n}_{\mathrm{c}}(X, \underline{\mathbf{C}})$$

sends 1 into $\mathrm{H}^{2n}_{\mathrm{c}}(X, \underline{\mathbf{C}})$. We may of course assume $X$ is connected without loss of generality. Then since the underlying topological space of $X$ is a smooth *connected* manifold, so $\mathrm{H}^{2n}_{\mathrm{c}}(U, \underline{\mathbf{C}}) \to \mathrm{H}^{2n}_{\mathrm{c}}(X, \underline{\mathbf{C}})$ is an isomorphism for any connected open neighborhood $U$ of the image point of $s$, our analytic claim for paracompact Hausdorff $X$ is *equivalent* to the special case when $X$ is the open unit $n$-ball over $A$ and $s$ is the 0-section. We conclude that it is necessary and sufficient to consider a *single* example in dimension $n$. Thus, we may consider $X = (\mathbf{P}^n_A)^{\mathrm{an}}$ and any $s \in X(A)$ (e.g., $s = [0, \dots, 0, 1]$). Looking back over the various equivalences through which we have passed, we are reduced to considering the *original* question of comparing $\mathrm{Tr}_f$ and

$\mathrm{Tr}_{f^{\mathrm{an}}}$ for $f : \mathbf{P}^n_A \to \mathrm{Spec}(A)$. But now everything is a base change from a situation over $\mathrm{Spec}(\mathbf{C})$, so by base change compatibility of $\mathrm{Tr}_f$ and $\mathrm{Tr}_{f^{\mathrm{an}}}$ we are reduced to the known case $A = \mathbf{C}$!

## 4. Chern classes on curves

Let $k$ be an algebraically closed field and let $X$ be a proper smooth connected curve over $k$. Using the canonical isomorphism $\mathrm{H}^1(X, \mathscr{O}_X^\times) \simeq \mathrm{Pic}(X)$ as in [C1, (B.4.1)] (and see the p. 284 remarks in [C2]), there is a canonical composite map

$$\mathrm{H}^1(X, \mathscr{O}_X^\times) \to \mathbf{Z} \to k \tag{4.1}$$

where the first step is determined by assigning to each line bundle $\mathscr{L}$ its degree. We will call (4.1) the *Chern class map* because in the analytic situation when $k = \mathbf{C}$ it is a classical fact that (4.1) coincides with the first Chern class of a line bundle (hence the title of this section), given by

$$\mathrm{H}^1(X, \mathscr{O}_X^\times) \to \mathrm{H}^2(X^{\mathrm{an}}, \underline{\mathbf{Z}}(1)) = \mathbf{Z},$$

where we use the coboundary map arising from the canonical exponential sequence

$$0 \to \underline{\mathbf{Z}}(1) \to \mathscr{O}_{X^{\mathrm{an}}} \stackrel{\exp}{\to} \mathscr{O}_{X^{\mathrm{an}}}^\times \to 1$$

on the compact Riemann surface $X^{\mathrm{an}}$. We will now prove that by using the abelian sheaf map

$$\mathrm{dlog} : \mathscr{O}_X^\times \to \Omega^1_{X/k}$$

defined by $f \mapsto \mathrm{d}f/f$, the Chern class map (4.1) is equal to the composite

$$\mathrm{H}^1(X, \mathscr{O}_X^\times) \to \mathrm{H}^1(X, \Omega^1_{X/k}) \stackrel{\mathrm{res}_{X/k}}{\longrightarrow} k, \tag{4.2}$$

with $\mathrm{res}_{X/k}$ equal to the residue map [C1, (B.1.2)] (also see the pp. 271, 275 remarks in [C2]).

The argument will ultimately reduce to the computation of the residue map in terms of Čech theory in [C1, p. 289]. It suffices to consider the case $\mathscr{L} = \mathscr{O}(x_1)$ for some $x_1 \in X(k)$, in which case we need to prove that (4.2) sends the class of $\mathscr{L}$ to $1 \in k$. Choose a rational function $t \in k(X)$ with a simple zero at $x_1$ and its only other zeros (if any) at some other point $x_0 \in X(k)$. In case $X$ has genus 0, we require that $t$ not have a pole at $x_0$. Thus, the finite non-empty pole set $P$ of $t$ is a subset of $X(k)$ which is disjoint from $\{x_0, x_1\}$. Let $V_0 = X - (P \cup \{x_0\})$, $V_1 = X - \{x_1\}$, so $\{V_0, V_1\}$ is an open affine covering of $X$ which trivializes $\mathscr{L}$: $\mathscr{L}|_{V_0}$ has a free basis $1/t$ and $\mathscr{L}|_{V_1}$ has a free basis 1. Thus, the Čech class in $\mathrm{H}^1(\{V_0, V_1\}, \mathscr{O}_X^\times)$ for $\mathscr{L}$ is represented by the unit $1/(1/t) = t$ on $V_0 \cap V_1$. Since

$$\mathrm{dlog}(t|_{V_0 \cap V_1}) = \mathrm{d}t/t \in \Omega^1_{X/k}(V_0 \cap V_1)$$

and the computation in [C1, p. 289] only requires that the *second* open set in the *ordered* open covering have complement consisting of a single point, we conclude that

$$\mathrm{res}_{X/k}(\mathrm{dlog}([\mathscr{L}])) = \mathrm{res}_{x_1}(\mathrm{d}t/t) = 1$$

since $t$ has a simple zero at $x_1$. This completes the proof of compatibility of the Chern class map and residue map for curves.

■

## References


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